Sobolev’s imbedding theorem in the limiting case with Lorentz space and BMO

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We consider the Gagliardo-Nirenberg type inequality in $\mathbb{R}^n$. Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. It is well known that the Sobolev space $H_0^{n/p,p}(\Omega)$, $1 < p < \infty$, is continuously embedded into $L^q(\Omega)$ for all $q$ with $p \leq q < \infty$. However, we cannot take $q = \infty$ in such an embedding. When $\Omega = \mathbb{R}^n$, Ogawa [11] and Ogawa-Ozawa [12] treated the Hilbert space $H^{n/2,2}(\mathbb{R}^n)$ and then Ozawa [15] gave the following general embedding theorem in the Sobolev space $H_n^{p,p}(\mathbb{R}^n)$ of the fractional derivatives which states that

$$
\|\Phi_p'(\alpha |u|^p)\|_{L^1(\mathbb{R}^n)} \leq C \|u\|^p_{L^p(\mathbb{R}^n)}
$$

(0.1)

holds for all $u \in H_n^{p,p}(\mathbb{R}^n)$ with $\|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leq 1$, where

$$
\Phi_p(\xi) := \exp(\xi) - \sum_{j=0}^{j_p-1} \frac{\xi^j}{j!} = \sum_{j=j_p}^{\infty} \frac{\xi^j}{j!}, \quad j_p := \min\{j \in \mathbb{N} \mid j \geq p - 1\}.
$$

The advantage of (0.1) gives the scale invariant form. In order to prove the above Trudinger type inequality, Ozawa [15] showed the following Gagliardo-Nirenberg type interpolation inequality which is equivalent to (0.1). For $1 < p < \infty$, there is a constant $M$ depending only on $n$ and $p$ such that

$$
\|u\|_{L^q(\mathbb{R}^n)} \leq Mq^{1/p'}\|u\|_{L^p(\mathbb{R}^n)}\|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)}^{1-p/q}
$$

(0.2)

holds for all $u \in H_n^{p,p}(\mathbb{R}^n)$ and for all $q$ with $p \leq q < \infty$. Our goals are the generalizations of (0.2) to the Gagliardo-Nirenberg type interpolation inequality with the Lorentz space and BMO. We shall state main theorems below.

**Theorem 0.1.** Let $1 < p_1 < \infty$.

(i) There exists a constant $C_{n,p_1}$ depending only on $n$ and $p_1$ such that

$$
\|u\|_{L^q} \leq C_{n,p_1} q^{1/r_2}\|u\|_{L^{p_1/p_2}}\|(-\Delta)^{n/(2r_1)}u\|_{L^{r_1/r_2}}^{1-p_1/q}
$$

(0.3)
holds for all \( u \in L(p_1, p_2) \) with \((-\Delta)^{n/(2r_1)}u \in L(r_1, r_2)\), where \( p_2, q, r_1 \) and \( r_2 \) are any numbers satisfying \( 1 \leq p_2 \leq p_1 \leq q < \infty \), \( p_1 \leq r_1 < \infty \) and \( 1 < r_2 < \infty \).

(ii) There exists a constant \( C_{n, p_1} \) depending only on \( n \) and \( p_1 \) such that

\[
\|u\|_{L^q} \leq C_{n, p_1} \frac{q^2}{q - p_1} \|u\|_{L(p_1, \infty)}^{p_1/q} \|(-\Delta)^{n/(2r_1)}u\|_{L(r_1, \infty)}^{1-p_1/q}
\]  

holds for all \( u \in L(p_1, \infty) \) with \((-\Delta)^{n/(2r_1)}u \in L(r_1, \infty)\), where \( q \) and \( r_1 \) are any numbers satisfying \( p_1 < q < \infty \) and \( p_1 \leq r_1 < \infty \).

We note that when we put \( p_1 = p_2 = r_1 = r_2 =: p \in (1, \infty) \) in (0.3), we can obtain (0.2) proved by Ozawa [15] immediately. Moreover, from Corollary 0.1, we obtain the Trudinger type inequalities equivalent to (0.3) and (0.4) as follows:

**Corollary 0.1.** Let \( 1 < p_1 < \infty \).

(i) For every \( 1 < r_2 < \infty \), there exists a constant \( C_{n, p_1, r_2} \) depending only on \( n, p_1 \) and \( r_2 \) such that the following holds. For arbitrary \( 0 < \alpha < C_{n, p_1, r_2} \), there exists a constant \( \tilde{C}_{n, p_1, r_2, \alpha} \) depending only on \( n, p_1, r_2 \) and \( \alpha \) such that

\[
\int_{\mathbb{R}^n} \Phi_{p_1, r_2} \left( \alpha \left( \frac{|u(x)|}{\|(-\Delta)^{n/(2r_1)}u\|_{L(r_1, r_2)}} \right)^{r_2} \right) dx \leq \tilde{C}_{n, p_1, r_2, \alpha} \left( \frac{\|u\|_{L(p_1, p_2)}}{\|(-\Delta)^{n/(2r_1)}u\|_{L(r_1, r_2)}} \right)^{p_1}
\]  

holds for all \( u \in L(p_1, p_2) \setminus \{0\} \) with \((-\Delta)^{n/(2r_1)}u \in L(r_1, r_2)\), where \( p_2, r_2 \) are any numbers satisfying \( 1 \leq p_2 \leq p_1 \leq r_1 < \infty \) and \( \Phi_{p_1, r_2} \) is defined by

\[
\Phi_{p_1, r_2}(\xi) := \sum_{j \geq 1} \frac{\xi^j}{j!} \quad \text{for } \xi \in \mathbb{R}.
\]

(ii) There exists a constant \( C_{n, p_1} \) depending only on \( n \) and \( p_1 \) such that the following holds. For arbitrary \( 0 < \alpha < C_{n, p_1} \), there exists a constant \( \tilde{C}_{n, p_1, \alpha} \) depending only on \( n, p_1 \) and \( \alpha \) such that

\[
\int_{\mathbb{R}^n} \tilde{\Phi}_{p_1} \left( \alpha \left( \frac{|u(x)|}{\|(-\Delta)^{n/(2r_1)}u\|_{L(r_1, \infty)}} \right) \right) dx \leq \tilde{C}_{n, p_1, \alpha} \left( \frac{\|u\|_{L(p_1, \infty)}}{\|(-\Delta)^{n/(2r_1)}u\|_{L(r_1, \infty)}} \right)^{p_1}
\]  

holds for all \( u \in L(p_1, \infty) \setminus \{0\} \) with \((-\Delta)^{n/(2r_1)}u \in L(r_1, \infty)\) and for all \( p_1 \leq r_1 < \infty \), where \( \tilde{\Phi}_{p_1} \) is defined by

\[
\tilde{\Phi}_{p_1}(\xi) := \sum_{j \geq 1} \frac{\xi^j}{j!} \quad \text{for } \xi \in \mathbb{R}.
\]
In fact, noting the definition of $\Phi_{p_1,r_2}$ (or $\tilde{\Phi}_{p_1}$), we exchange the integral of (0.5) (or (0.6)) for the sum, and then by applying (0.3) (or (0.4)) for each integral, we have the Trudinger type inequality.

By putting $q_1 = q_2$ in Theorem 0.2, we have the following Corollary 0.2.

**Theorem 0.2.** (i) For every $1 \leq p_1 < \infty$, there exists a constant $C_{n,p_1}$ depending only on $n$ and $p_1$ such that

$$
\|u\|_{L^q} \leq C_{n,p_1} \|u\|_{L(p_1,p_2)}^{p_1/q} \|u\|_{BMO}^{1-p_1/q}
$$

holds for all $u \in L(p_1,p_2) \cap BMO$, where $p_2$ and $q$ are any numbers satisfying $1 \leq p_2 \leq p_1 \leq q < \infty$.

(ii) For every $1 < p_1 < \infty$, there exists a constant $C_{n,p_1}$ depending only on $n$ and $p_1$ such that

$$
\|u\|_{L^q} \leq C_{n,p_1} \frac{q^2}{q-p_1} \|u\|_{L(p_1,\infty)}^{p_1/q} \|u\|_{BMO}^{1-p_1/q}
$$

holds for all $u \in L(p_1,\infty) \cap BMO$ and for all $p_1 < q < \infty$.

Moreover, from Corollary 0.2, we obtain the Trudinger type inequalities equivalent to (0.7) and (0.8) as follows:

**Corollary 0.2.** (i) For every $1 \leq p_1 < \infty$, there exists a constant $C_{n,p_1}$ depending only on $n$ and $p_1$ such that the following holds. For arbitrary $0 < \alpha < C_{n,p_1}$, there exists a constant $\tilde{C}_{n,p_1,\alpha}$ depending only on $n$, $p_1$ and $\alpha$ such that

$$
\int_{\mathbb{R}^n} \Phi_{p_1}(\alpha \frac{|u(x)|}{\|u\|_{BMO}}) \, dx \leq \tilde{C}_{n,p_1,\alpha} \left( \frac{\|u\|_{L(p_1,p_2)}}{\|u\|_{BMO}} \right)^{p_1}
$$

holds for all $u \in L(p_1,p_2) \cap BMO \setminus \{0\}$ and for all $1 \leq p_2 \leq p_1$, where $\Phi_{p_1}$ is defined by

$$
\Phi_{p_1}(\xi) := \sum_{j \leq p_1 \atop j \in \mathbb{N}} \frac{\xi^j}{j!} \quad \text{for } \xi \in \mathbb{R}.
$$

(ii) For every $1 < p_1 < \infty$, there exists a constant $C_{n,p_1}$ depending only on $n$ and $p_1$ such that the following holds. For arbitrary $0 < \alpha < C_{n,p_1}$, there exists a constant $\tilde{C}_{n,p_1,\alpha}$ depending only on $n$, $p_1$ and $\alpha$ such that

$$
\int_{\mathbb{R}^n} \tilde{\Phi}_{p_1}(\alpha \frac{|u(x)|}{\|u\|_{BMO}}) \, dx \leq \tilde{C}_{n,p_1,\alpha} \left( \frac{\|u\|_{L(p_1,\infty)}}{\|u\|_{BMO}} \right)^{p_1}
$$

holds for all $u \in L(p_1,\infty) \cap BMO \setminus \{0\}$, where $\tilde{\Phi}_{p_1}$ is defined by

$$
\tilde{\Phi}_{p_1}(\xi) := \sum_{j > p_1 \atop j \in \mathbb{N}} \frac{\xi^j}{j!} \quad \text{for } \xi \in \mathbb{R}.
$$
Finally, we shall state the application to the Brezis-Gallouet-Wainger type inequality. In fact, from Corollary 0.1 (i), we can obtain the inequality as follows:

**Theorem 0.3.** For every $1 < p_1 < \infty$, $1 \leq q \leq \infty$ and $n/q < m < \infty$, there exists a constant $C_{n,p_1,q,m}$ depending only on $n$, $p_1$, $q$ and $m$ such that

$$
\|u\|_{L^\infty} \leq C_{n,p_1,q,m} \left[ 1 + \left( \| u \|_{L(p_1,p_2)} + \| (-\Delta)^{n/(2r_1)} u \|_{L(r_1,r_2)} \right) \times \left( \log (e + \| (-\Delta)^{m/2} u \|_{L^q}) \right)^{1/r_2'} \right]
$$

(0.9)

holds for all $u \in L(p_1,p_2)$ with $(-\Delta)^{n/(2r_1)} u \in L(r_1,r_2)$ and $(-\Delta)^{m/2} u \in L^q$, where $p_2$, $r_1$ and $r_2$ are any numbers satisfying $1 \leq p_2 \leq p_1 \leq r_1 < \infty$ and $1 \leq r_2 < \infty$.

we note that the inequality with $p_1 = p_2 = r_1 = r_2 =: p \in (1,\infty)$ in (0.9) coincides with the classical Brezis-Gallouet-Wainger inequality.

**References**


