THE HELMHOLTZ DECOMPOSITION OF SOME BANACH SPACES IN SPECIAL SECTORIAL DOMAINS

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In this talk we give an explicit representation formula for the Helmholtz projection from some Banach space $V(\Omega)$ to its solenoidal subspace, where $\Omega$ is a direct product of special sectorial plains, half lines and whole lines. More precisely, suppose that $\Omega$ is a domain of the form

$$\Omega := \Omega_{m_1} \times \cdots \times \Omega_{m_n} \times (\mathbb{R}_+)^n_2 \times \mathbb{R}^{n_3} \subset \mathbb{R}^{2n_1+n_2+n_3},$$

where

$$\Omega_m := \left\{ x = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2 : r > 0, 0 < \theta < \frac{\pi}{m} \right\} \quad \text{for} \quad m \in \mathbb{N}$$

and

$$\mathbb{R}_+ := \{ x \in \mathbb{R} : x > 0 \}.$$

Let

$$V(\Omega) := \{ f \in \mathcal{D}'(\Omega) : \text{there is } g \in V(\mathbb{R}^n) \text{ s.t. } g|_\Omega = f \},$$

$$\| f : V(\Omega) \| := \inf_{g \in V(\mathbb{R}^n), g|_\Omega = f} \| g : V(\mathbb{R}^n) \|.$$

Note that $V(\Omega)$ is also Banach space. Let us assume that $V(\Omega)$ and $V(\mathbb{R}^n)$ have the following properties.

(i) $\mathcal{C}_\text{comp}^{\infty}(\Omega)^b_{\|V(\cdot)\|} = V(\Omega)$.

(ii) $\| \varphi : V(\mathbb{R}^n) \| = \| \varphi : V(\Omega) \|$ for $\varphi \in \mathcal{C}_\text{comp}^{\infty}(\Omega)$.

(iii) Let the operator $R_k$ be the Riesz operator of a kernel $\frac{x_k}{|x|^{n+1}}$, then

$$\| R_k f : V(\mathbb{R}^n) \| \leq C \| f : V(\mathbb{R}^n) \|,$$

for $1 \leq k \leq n$.

(iv) Let $\eta x := Ax + b$, $A \in O(n)$, $b$ be some vector and $O(n)$ be the set of $n \times n$ orthogonal matrices, then

$$C^{-1} \| f : V(\mathbb{R}^n) \| \leq \| \eta f : V(\mathbb{R}^n) \| \leq C \| f : V(\mathbb{R}^n) \|.$$

(v) Let $V^*(\Omega)$ be a dual space of $V(\Omega)$. The dual space $V^*(\Omega)$ also has properties (i)–(iv).

For example, some amalgam spaces and some Orlicz spaces (see [7] and [14]) satisfy these properties. (Note that the property of (iii) in amalgam space is proved by [12]).

Let us define

$$X_V(\Omega) := \{ \vec{u} \in \mathcal{C}^\infty(\Omega) \cap C(\bar{\Omega}) \cap V(\Omega) : \text{div} \vec{u} = 0, \vec{u} \cdot \vec{n}|_\Omega = 0 \},$$
$X_p(\Omega)$ is the closure of $X_V(\Omega)$ in $V(\Omega)$-norm,

$$Y_V(\Omega) := \{ \nabla q : q \in C^\infty(\Omega), \ \nabla q \in V(\Omega) \cap C(\overline{\Omega}) \}$$

and $Y_V(\Omega)$ is the closure of $Y_V(\Omega)$ in $V(\Omega)$, where $\vec{n}$ is the unit exterior normal vector of $\partial \Omega$. The aim of this talk is to give an explicit integral kernel $E^*$ for $P$ of the form

$$P(\vec{u})(x) := \vec{u} + \nabla_x \int \Omega \nabla_z E^*(x, z) \cdot \vec{u}(z) dz,$$

where $P : V(\Omega) \to X_V(\Omega)$ is the Helmholtz projection which gives the direct sum decomposition:

$$V(\Omega) = X_V(\Omega) \oplus Y_V(\Omega).$$

**References**


