Introduction to Joseph's theory on orbital varieties

We explain Joseph's theory on orbital varieties.

For a semi-simple Lie algebra $\mathfrak{g}$ (over the field of complex numbers), we have its enveloping algebra $U(\mathfrak{g})$. Since $U(\mathfrak{g})$ is a non-commutative ring, we have classification problems of nice $U(\mathfrak{g})$-ideals arising from the context of general theory of non-commutative rings.

One important class of such $U(\mathfrak{g})$-ideals is primitive ideals. It is defined by purely ring-theoretic way, but has a concrete description in terms of highest weight modules of $U(\mathfrak{g})$ (Duflo's surjectivity theorem). This enables us to treat primitive ideals in terms of the charactor of $Z(\mathfrak{g})$ (the center of $U(\mathfrak{g})$) and the Weyl group W of $\mathfrak{g}$.

A primitive ideal of $U(\mathfrak{g})$ carries an invariant called its Goldie rank, which measures the size of the quotient ring in some (non-commutative) way. A general method in the study of semi-simple Lie algebras (translation principle) allows us to divide the set of primitive ideals into infinite families in which the Goldie ranks behave as polynomials (depending on characters of $Z(\mathfrak{g})$).

Associated to $\mathfrak{g}$, we have its subvariety $\mathcal{N}$ given by the set of nilpotent elements. We call $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$. (Here we regard $\mathfrak{g}$ as an algebraic variety isomorphic to $\mathbb{C}^{\dim \mathfrak{g}}$.) Joseph defined certain collection of subvarieties of $\mathcal{N}$, which is usually called orbital varieties.

An orbital variety, together with its embedding into $\mathfrak{g}$, defines its characteristic polynomials (usually called the Joseph polynomials). The span of Joseph polynomials admits W-action, which is a reincanation of the Springer representation of W.

The (first) main result of Joseph's theory is the identification of some span of Joseph polynomials and that of Goldie rank polynomials. (It further gives an equality of special Joseph polynomials and special Goldie rank polynomials up to scalar when $\mathfrak{g}$ is of type A.) Moreover, there is a close relationship between the inclusion relation of primitive ideals and the inclusion relation of orbital varieties (at least when $\mathfrak{g}$ is of type A), which is still a subject of on-going studies. (For detailed exposition, see Joseph [Perspectives in Math. **17** 53–99, Academic Press, 1997] §1–4 and Hinich-Joseph [Selecta Math. **11** 9–36 (2005)] §1 and 6.)

We explain some introductory part of this story as follows:

1. We introduce the basic notion from general theories, including equivariant K-theory, characteristic polynomials, and nilpotent cones;
2. We define orbital varieties and Joseph polynomials and see why it admits an action of the Weyl group;
3. We explain
   - (a) the statement of Dulfo's surjectivity theorem;
   - (b) the definition of the Goldie rank polynomials;
   - (c) relation of the Joseph polynomials and the Goldie rank polynomials;

We might omit some definitions if it is already familiar in the other lectures.