Invariant Hilbert spaces of holomorphic functions on homogeneous Kähler manifolds

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Let M be a Kähler manifold on which a Lie group G acts transitively as Kähler automorphisms. When the Kähler form on M is integral, we have a canonical holomorphic Hermitian line bundle L, called *quantization bundle*, on M. Then the group G acts on L naturally, so that a continuous representation ρ of G is defined on the space $\Gamma_{hol}(L)$ of holomorphic sections of L, where $\Gamma_{hol}(L)$ is regarded as a topological vector space with the compact-open topology. We shall consider the condition that ρ is unitarizable, that is, there exists a Hilbert space $\mathcal{H}(L) \neq \{0\}$ continuously imbedded into $\Gamma_{hol}(L)$ such that $(\rho, \mathcal{H}(L))$ is a unitary representation of G. If the space $\Gamma^2_{hol}(L)$ of square integrable holomorphic sections of L is non-trivial, then we can take $\Gamma^2_{hol}(L)$ as the Hilbert space $\mathcal{H}(L)$. On the other hand, even if $\Gamma^2_{hol}(L) = \{0\}$, there may exist a non-trivial $\mathcal{H}(L)$ giving us a unitary representation of G. When G is non-compact and reductive, then $(\rho, \Gamma^2_{hol}(L))$ is nothing but the holomorphic discrete series representation, while $(\rho, \mathcal{H}(L))$ is a realization of the unitarizable highest weight representation.

The fundamental theorem of homogeneous Kähler manifold states that M has a structure of holomorphic fiber bundle over a homogeneous bounded domain in which the fiber is the product of a flat homogeneous Kähler manifold and a compact simply connected homogeneous Käler manifold. At the same time, the Lie algebra of G has a specific structure called *Kähler algebra*, which is finely studied by several authors. It is natural to try to apply these geometric and algebraic results to the study of the representation $(\rho, \mathcal{H}(L))$ of G. In this lecture, I present a general framework of the problem, and give results for the case that M is a homogeneous bounded domain.

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