# On the blow up phenomenon for the $L^2$ critical non linear Schrödinger Equation

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This series of lectures will be devoted to the study of the singularity formation for the $L^2$ critical non linear Schrödinger equation

$$(NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases} \tag{1}$$

with $u_0 \in H^1 = \{u, \nabla u \in L^2(\mathbf{R}^N)\}$ in dimension $N \geq 1$. This equation for $N = 2$ appears in physics as a universal model to describe self trapping of waves propagating in non linear media. The physical expectation for large smooth data is the concentration of part of the $L^2$ mass in finite time corresponding to the focusing of the laser beam. If some explicit examples of this phenomenon are known, and despite a number of both numerical and mathematical works, a general description of blow up dynamics is mostly open.

(NLS) is an infinite dimensional Hamiltonian system with energy space $H^1$ without any space localization property. It is in this context together with the critical generalized KdV equation the only example where blow up is known to occur. For (NLS), an elementary proof of existence of blow up solutions is known since the 60's but is based on energy constraints and is not constructive. In particular, *no qualitative information of any type on the blow up dynamics is obtained this way.*

The natural questions we address regarding blow up dynamics in the energy space are the following:
-Does there exist a Hamiltonian characterization of blow up solutions, or at least necessary conditions for blow up simply expressed from the Hamiltonian invariants?
-Assuming blow up, does there exist a universal blow up speed, or are there several possible regimes? Among these regimes, which ones are stable?
-Does there exist a universal space time structure for the formation of singularities independent at the first order of the initial data?

Considering first a more general class of non linear Schrödinger equations, we will recall in the subcritical cases the proof of global wellposedness of the problem. We will then restrict ourselves to the study of the critical case (1) and review the main results concerning the description of blow up dynamics obtained by various authors in the 80's and 90's. We will then focus onto the presentation and the proof of some of the recent results obtained in collaboration with Frank Merle in the setting of a perturbative analysis close to the exceptional solution to (1): the ground state solitary wave. The presentation will be mostly self contained provided the prior knowledge of some standard tools in the study of non linear PDE's.