Research Report

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I am interested in equivariant symplectic geometry. I have classified complexity one Hamiltonian actions of rank one compact Lie groups. This strengthens our knowledge of the general classification for symplectic manifolds with group actions. I have then applied this work to construct new examples of Lagrangian submanifolds.

1 Hamiltonian group actions

Let (M, ω) be a symplectic manifold and G be a compact connected Lie group that acts effectively on M by symplectic transformations. A **moment map** $\Phi\colon M \to \mathfrak{g}^*$ is a G-equivariant map such that for every ξ in the Lie algebra $\mathfrak{g}$ of G,

$$\iota(\xi_M)\omega = -d\langle \Phi, \xi \rangle$$

where $\xi_M\colon M \to TM$ denotes the induced vector field of ξ on M. If there is a moment map, we say that the action is **Hamiltonian**.

For a point α in the dual of the Lie algebra of G, the **reduced space** at α is the symplectic space $M_\alpha = \Phi^{-1}(G \cdot \alpha)/G = \Phi^{-1}(\alpha)/G_\alpha$, where $G \cdot \alpha$ is the coadjoint orbit through α and G_α denotes the stabilizer of α.

Six dimensional manifolds equipped with Hamiltonian SU(2) or SO(3) actions are so-called complexity one spaces since their generic reduced spaces are two dimensional. In [C1], I provide a complete set of invariants to classify these spaces up to equivariant symplectomorphisms:

Theorem 1.1. *Let G be* SU(2) *or* SO(3)*. Let (M, ω, Φ) and (M', ω', Φ') be compact connected six dimensional Hamiltonian G-manifolds such that $\Phi(M) = \Phi'(M)$. Then M and M' are equivariant symplectomorphic if and only if they have the same Duistermaat-Heckman function, the same genus, the same isotropy skeleton, the same principal stabilizers of the manifolds and of the zero level sets, and the same first Stiefel-Whitney class of the zero level sets when applicable.*

2 Lagrangian embeddings

Let (M, ω) be a symplectic manifold. A submanifold $L \subset M$ is called **Lagrangian** if $\dim L = \frac{1}{2} \dim M$ and ω vanishes on $T(L)$.

It is known that every orbit in the preimage of zero under a moment map is isotropic. Applying this fact to various nonabelian Hamiltonian actions, I have discovered interesting examples that provide a test ground to further study different aspects of Lagrangian submanifolds [C2, C3]:

Proposition 2.1. *There exists a Lagrangian submanifold of $\mathbb{C}P^3$, which is a quotient of $\mathbb{R}P^3$ by the dihedral group D_3.*

Proposition 2.2. *Denote by $G_2^+(\mathbb{R}^{n+2})$ the real Grassmannian of oriented* 2*-planes in $\mathbb{R}^{n+2}$. There exists a Lagrangian sphere S^n in $G_2^+(\mathbb{R}^{n+2})$.*

3 Current projects

3.1 Contact geometry

The convexity properties of the moment map for Hamiltonian group actions, and the connectedness of the fibers of the moment map have numerous applications in symplectic geometry. For example, it is used in the classification of Hamiltonian G-manifolds, in geometric quantization, and in the study of the existence of invariant Kähler structure.

There is a notion of contact moment map for an action of a group on a contact manifold. Lerman [Le] has proved that the image cone of a moment map for an action of a torus on a contact compact connected manifold is a convex polyhedral cone and that the moment map has connected fibers provided the dimension of the torus is bigger than 2 and that no orbit is tangent to the contact distribution.

In a joint project with Nan-kuo Ho, we try to determine if there exists a convexity property for nonabelian actions on contact manifolds.

3.2 Symplectic surgery

The symplectic cutting can be an effective tool to construct compact manifolds out of noncompact ones. The only setback of cutting is that it may introduce orbifold singularities. This is the result of taking the quotient by the S^1 action, which may have finite but nontrivial stabilizers. I try to investigate symplectic surgeries to resolve these singularities.

3.3 Cotangent bundles

The real Grassmannian can be decomposed into a union of the cotangent bundle of the Lagrangian sphere and a symplectic manifold. For an arbitrary $n \geq 2$, we have

$$G_2^+(\mathbb{R}^{n+2}) \cong T^*S^n \cup G_2^+(\mathbb{R}^{n+1}).$$

Naively, starting from a Lagrangian, we would like to compactify its cotangent bundle by adding a symplectic manifold. We would further study if this decomposition helps us understand the relation between the topology of the manifolds and the Lagrangian submanifolds.

References

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[BC1] P. Biran & K. Cieliebak, *Symplectic Topology on subcritical manifolds.* Comment. Math. Helv. **76** (2001), 712–753

[C1] R. Chiang, *Complexity one Hamiltonian* SU(2) *and* SO(3) *actions.* To appear in Amer.J.Math.

[C2] R. Chiang, *New Lagrangian submanifolds of* $\mathbb{C}\mathrm{P}^n$. Preprint: math.SG/0303262.

[C3] R. Chiang, *Lagrangian spheres.* In preparation.

[HNP] J. Hilgert, K-H. Neeb and W. Plank, *Symplectic convexity theorems and coadjoint orbits.* Compositio Math. **94** (1994), no. 2, 129–180.

[Le] E. Lerman, *A convexity theorem for torus actions on contact manifolds.* Illinois J. Math. **46** (2002), no. 1, 171–184.

[LMTW] E. Lerman, E. Meinrenken, S. Tolman and C. Woodward, *Nonabelian convexity by symplectic cuts.* Topology **37** (1998), no. 2, 245–259.

List of Publications

1. Complexity one Hamiltonian SU(2) and SO(3) actions, to appear in American Journal of Mathematics, math.SG/0301168
2. New Lagrangian submanifolds of $\mathbb{C}\mathrm{P}^n$, submitted, math.SG/0303262

Presentations

1. Mar. 2004 **Noncommutative Geometry and Physics 2004**, Keio University
2. Feb. 2004 **Symposium**, Hokkaido University
3. Nov. 2003 **Colloquium**, Academia Sinica, Taiwan
4. Sep. 2003 **CCS Geometry Seminar**, Academia Sinica, Taiwan
5. Aug. 2003 **Exposé**, Université Louis Pasteur, Strasbourg, France
6. Feb. 2003 **Colloquium**, Instituto de Matemáticas Unidad Morelia UNAM, Mexico
7. Nov. 2002 **Symplectic Seminar**, University of Toronto, Canada
8. Jun. 2002 **Geometry Seminar**, Univeristy of Haifa, Israel
9. Mar. 2002 **Topology and Geometry Seminar**, Hebrew University of Jerusalem, Israel
10. Nov. 2000 **Geometry and Topology RAP**, University of Illinois at Urbana-Champaign
11. Apr. 2000 **Symplectic Geometry Seminar**, University of Illinois at Urbana-Champaign
12. Dec. 1999 **Symplectic Geometry Seminar**, University of Illinois at Urbana-Champaign