1. Description of research

1.1. **Conflict sets.** Let M_i $1 \leq i \leq l$ be l compact manifolds of codimension one in a smooth ambient manifold X, embedded by $\gamma_i \colon M_i \to X$. Finsler metrics on TX are homogeneous Hamiltonians on T^*X. Assume that there are given $H_1, \cdots, H_l$ Hamiltonians on T^*X, such that all the Finsler metrics are complete and without conjugate or cut-loci. Then we have l distance functions d_i, $1 \leq i \leq l$. Each of these gives phase functions

$$F \colon M_i \times X \to \mathbf{R}$$

on each of the M_i namely $F(x, s_i) = d_i(\gamma_i(s_i), x)$. When for some $x \in \mathbf{R}^n$ there are critical points $s_i^0 \in M_i$ of the functions $f_{i,x}$ all having one and the same critical value x is said to lie on the conflict set of the M_i. Thus the conflict set is the set of points x that lie at equal distance from a number of submanifolds.

Conflict sets are a generalization of two concepts: medial axis and Voronoi diagram. These last two are widely used in applications. The medial axis is an invariant of shape. It is the most suitable skeleton of a spatial shape for computer storage and manipulation. When in a Voronoi diagram the point sites are replaced by compact submanifolds one obtains a subset of the conflict set of the submanifolds. Such Voronoi diagrams are used in motion planning for robots that need to avoid obstacles.

Fix the H_i. For generic embeddings γ_i of the M_i the conflict set is the projection of a conic Lagrangian submanifold of $T^*\mathbf{R}^n$. Moreover if $n - l \leq 4$ then there are up to a local diffeomorphism in $\mathbf{R}^n$ a finite number of normal forms of conflict sets. If $l > 2$ then these normal forms are conic Lagrangian manifolds of higher type. Therefore the theorem tells us what the "nice" dimensions for conflict sets are.

In the proof of the theorem as an added bonus we get a method to construct a conic Lagrangian manifold of higher type, using Lagrange multipliers.

1.2. **1-parameter families of conflict sets.** I obtained a complete list of singularities one can expect in a generic one-parameter family of conflict sets in the planar case. There is, as was to be expected, a relation between this list of singularities and singularities of 2-parameter families of fronts in the plane. In particular, dangerous self-tangencies, are unavoidable in such families of conflict sets. There is a duality between these and curves on a torus : the pre-image of the conflict set. Hyperbolic Morse singularities correspond to dangerous self-tangencies. I explain what are the relations between the list of singularities in this geometric setting and in other related settings, such as symmetry sets.

1.3. **On cut and conjugate loci.** I have developed techniques to calculate numerically the conjugate locus. These are based on the following differential equation for geodesics on a smooth surface $\{F = 0\}$ in $\mathbf{R}^n$:

$$\dot{x} = v \quad \dot{v} = -\frac{\frac{\partial^2 F}{\partial x^2} vv}{\|\frac{\partial F}{\partial x}\|^2} \frac{\partial F}{\partial x} \quad (x, v) \in T\,\mathbf{R}^n$$

Using Cartan's characterization of Jacobi fields as the derivative of the geodesic flow I derived a similar equation for the Jacobi fields.

1.4. **Morse functions on Delaunay triangulations.** In applied mathematics the finite element method is an important tool to solve PDEs. To apply the method one needs to triangulate the domain. A cheap way to get such a triangulation is the Delaunay triangulation, associated to a point set $\{P_1, \cdots, P_N\} \subset \mathbf{R}^n$. It is well known that in such triangulations slivers arise, they are very flat tetrahedra. Slivers are very bad for the finite element method. Therefore it is useful to study the structure of the triangulation. For this I introduced in a joint article with D. Siersma the Morse poset.

For a point set $\{P_1, \cdots, P_N\} \subset \mathbf{R}^n$, consider the sets $K_i \subset \mathbf{R}^n$, the closure of the set of those $x \in \mathbf{R}^n$ that have P_i as closest point. The K_i and their intersections form the Voronoi diagram. The nerve of the set of sets $\{K_i\}_{1\leq i\leq N}$ is a cell complex dual to the Voronoi diagram. It is called the Delaunay triangulation. The Delaunay triangulation is not always a triangulation, as in the case where the point set are the vertices of a rectangle.

Now assign a weight w_i to each of the points. Set $g_i(x) = \|x - P_i\|^2 - w_i$ and $g(x) = \min_i g_i(x)$ to get a topological Morse function. We also get a divison of $\mathbf{R}^n$ in polyhedra by putting, for each $\alpha \subset \{P_1, \cdots, P_N\}$, $\mathrm{Pow}(\alpha) \subset \mathbf{R}^n$ to be the closure of

$$\{x \in \mathbf{R}^n \mid g(x) = g_i(x) \text{ if } i \in \alpha \ g(x) < g_i(x) \text{ if } i \notin \alpha\}$$

Such a division of $\mathbf{R}^n$ is called a power diagram, or additively weighted Voronoi diagram. Dual to it lies the coherent triangulation. The critical points of index i of g can uniquely be identified with simplices of dimension i in the coherent triangulation. The simplices that have a critical point of g in their interior form the Morse poset.

When the $\{P_1, \cdots, P_N\} \subset \mathbf{Z}^n$ its skeleton of $n-1$-dimensional components is also known as the tropical hypersurface, or the spine of the amoeba of an algebraic variety in $\mathbf{C}^{*n}$. This can be seen as follows. Instead of $g(x)$ use $f(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}g(x)$ and $f_i(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}g_i(x)$. We put $f(x) = \max_i f_i(x)$. Then

$$\mathrm{Pow}(\alpha) = \text{Closure of } \{x \in \mathbf{R}^n \mid f(x) = f_i(x) \text{ if } i \in \alpha \ f(x) > f_i(x) \text{ if } i \notin \alpha\}$$

Thus there is a strong link between computational and tropical geometry.

It is possible to define a so-called discrete Morse function on a coherent triangulation. A discrete Morse function is, roughly speaking, a function on a simplicial complex whose adjacent level sets are either related by a simplicial collapse (regular values) or by an attachment of a cell (critical values). I proved: There exists a discrete Morse function on the coherent triangulation that is a discretization of g. So the Morse poset fits in nicely with current research in combinatorics.

1.5. **The medial axis is of moment map.** The medial axis can also be described using tropical geometry. The medial axis is a corner locus, the set of points of a convex function where the function is not differentiable. Let $\gamma \colon M \to \mathbf{R}^n$ be the embedding of a compact manifold without boundary M. Define a family of functions

$$f_s(x) = \langle x, \gamma(s)\rangle - \frac{1}{2}\|\gamma(s)\|^2 = \frac{1}{2}\|x\|^2 - \frac{1}{2}\|x - \gamma(s)\|^2$$

and take its maximum over all s:

$$f(x) = \max_{s \in M} f_s(x) \tag{1}$$

Then we have $\frac{1}{2}\|x\|^2 - f(x) = \min_{s\in M} \frac{1}{2}\|x-\gamma(s)\|^2$. I proved:

$$f(x) = \lim_{h\to\infty} \log_h \left(\int_M h^{\langle x,\gamma(s)\rangle - \frac{1}{2}\|\gamma(s)\|^2} ds \right) \tag{2}$$

and the corner locus of f is the medial axis of γ.

The identity (2) holds because of what is known as Maslov dequantization in tropical geometry:

$$\max(a,b) = \lim_{h\to\infty} \log_h (h^a + h^b)$$

Moreover it is inspired by the moment map from toric geometry. For a point set we put (as in the above)

$$f_i(x) = \langle x, P_i\rangle - \frac{1}{2}\|P_i\|^2 + \frac{1}{2}w_i = \langle x, P_i\rangle + c_i \quad 1 \le i \le N$$

The functions

$$H_h(x) = \frac{\sum_{i=1}^N h^{f_i(x)} P_i}{\sum_{i=1}^N h^{f_i(x)}}$$

are diffeomorphisms from $\mathbf{R}^n$ to $\mathrm{CH}(\{P_1, \cdots, P_N\})$. The functions $H_h(x)$ are of course the moment maps from toric geometry. We also have

$$H_h(x) = \frac{\partial}{\partial x}\left(\log_h \left(\sum_{i=1}^N h^{f_i(x)} \right)\right)$$

Passing to the limit with infinitely many points on M and letting $h \to \infty$ we see that the generalized Clarke derivative of $f(x)$ in (1) at points on the medial axis is the support function of the convex hull of those $s \in M$ for which $\frac{1}{2}\|x-\gamma(s)\|^2$ achieves its non-unique global minimum.

2. Publications and preprints

- M. van Manen, *From the cut-locus, via the medial axis, to the Voronoi diagram and back.*, Hokkaido University technical report series in Mathematics, #99, 2005, pp. 1-40, revised version submitted as *Maxwell strata and caustics*
- M. van Manen, D. Siersma, *Power diagrams and their applications*, arXiV:math.MG/0508037, 2004, pp. 1-23
- M. van Manen, *On dangerous self-tangencies in families of conflict sets*, Hokkaido University preprint series in Mathematics #703, 2005, pp. 1-13,
- M. van Manen, D. Siersma, *The nine generic tetrahedra*, arXiV:math.MG/0410251, 2004, pp. 1-14
- Martijn van Manen, *Conflict sets, orthotomics, pedals and billiards as canonical relations*, 2004, *To be published in Math. Proc. Cambridge Philos. Soc.*
- M. van Manen *The geometry of conflict sets*, Rijksuniversiteit te Utrecht, Utrecht, 2003, Dissertation, Universiteit Utrecht, Utrecht, 2003. pp. x+100, ISBN 90-393-3416-1, MR 2000001
- M. van Manen, *Curvature and torsion formulas for conflict sets*, Geometry and topology of caustics – CAUSTICS 2002 (Warsaw), pp. 209-222, 2003, Polish Acad. Sci. , MR 2056442
- M. van Manen, *The geometry of conflict sets*, Rijksuniversiteit te Utrecht, Utrecht, 2003, Dissertation, Universiteit Utrecht, Utrecht, 2003. pp. x+100, ISBN 90-393-3416-1, MR 2000001
- Rachel Brouwer, Thijs Brouwer, Cor Hurkens, Martijn van Manen, Carolynne Montijn, Jan Schreuder, and J. F. Williams, *Magma design automation: component placement on chips; the "holey cheese" problem*, Proceedings of the Forty-Second European Study Group with Industry (Amsterdam, 2002) (Amsterdam), CWI Syllabi, vol. 51, Math. Centrum Centrum Wisk. Inform., 2002, pp. 77–90.

3. Talks and conference visits

- Kumamoto university, differential geometry seminar, February 2006
- Kitami institute of technology, introductory talk on oriented matroids, February 2006
- Singularities of differential systems, Hokkaido University, October 2005
- Advanced School and workshop on singularity theory, ICTP Trieste, August 2005 (talk and poster) COE lectures, Hokkaido university, June 2005 (3 talks)
- Singularity seminar Hokkaido university, May 2005.
- Around primitive forms, Kyoto RIMS, January 2005 (no talk)
- Symposium on mathematical aspects of image processing and computer vision 2004, Hokkaido university, November 2004
- Hakodate singularity theory conference. September 2004.
- Singularity seminar Hokkaido university, June 2004.
- Jussieu, séminaire sur les singularités. March 2004.
- Séminar "Mathematical challenges from the life sciences", Oberwolfbach, November 2003 (no talk).
- Warwick University, November 2003.
- Stafcolloquium Universiteit Utrecht, June 2003.
- Caustics 2002, May 2002, Banach Center, Warsaw.
- Arnol'd Seminar, September 2001, Lomonosov University, Moscow.
- Singularity theory and its applications to geometry, December 2000, University of Liverpool.
- Singularity theory and its applications to wave propagation and dynamical systems, September 2000, Newton Institute Cambridge.