RESEARCH REPORT

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1. Research Activities

The subjects of my researches in 2004 are concerned with several problems issued in Partial Differential Equations and Harmonic Analysis.

1.1. **Partial Differential Equations.** I studied dispersive equation (mainly Schrödinger type equation) and Navier-Stokes equation for the fluids with vacuum.

1.1.1. *Linear dispersive equation.* (1) The first problem concerned is to prove that the solution $u(x,t) = e^{it(-\Delta)^{\frac{a}{2}}} f(x)$ of linear dispersive equation $iu_t + (-\Delta)^{\frac{a}{2}} u = 0, u(0) = f$ has the mapping properties that $\| \sup_{t\in\mathbb{R}} |u(\cdot,t)| \|_{L^p} \lesssim \|f\|_{H^{\frac{1}{4}}}$ for any $a > 0 (\neq 1)$ and for some $p \geq 2$.

This problem was initiated by L. Carleson for $a = 2$ [3] and extensively studied by many people, for instance see [2, 12–15, 17–21]. But it remains still open. My collaborator and I (hereafter we) could prove partially that this is true for functions of finite linear combination of radial and spherical harmonic functions by using asymptotic behavior of Bessel functions and a boundedness of one dimensional oscillatory integrals [12, 14]. Furthermore, generalizing the operator $(-\Delta)^{\frac{a}{2}}$, we could extend the convergence problem to $e^{it\varphi((-\Delta)^{\frac{1}{2}})} f$ for some function φ having zeros and singularity on the sphere [13].
(2) The second is to determine the pair $(q, s, r, \tilde{q}, \tilde{r})$ satisfying $\| \int_0^t e^{i(t-s)(-\Delta)^{\frac{a}{2}}} F(\cdot, s)\, ds \|_{L_t^{q,\,s} L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}',\,2} L_x^{\tilde{r}'}}$, where $L^{q,\,s}$ is the Lorentz space.

The estimate above (*Strichartz estimate*) is a very useful tool, for example, to the existence of nonlinear dispersive equations. For the cases of $a = 1$ or 2, many researchers obtained several Strichartz estimates on Lebesgue space($L^q L^r$)(see for instance [21]). In this connection, we could characterize the estimates with respect to the value of a and extend the Lebesgue estimate to Lorentz estimate ($L^{q,s}L^r$) [1]. We proved in particular that if $a = 2$, then the above Strichartz estimate holds for all pairs (q, s, r) and $(\tilde{q}, \tilde{r})$ such that $\frac{1}{q} + \frac{n}{2r} = \frac{n}{4}$, $\frac{1}{\tilde{q}} + \frac{n}{2\tilde{r}} = \frac{n}{4}$ and $2 < s \leq q$. This estimate cannot be simply achieved by an interpolation (see Remark 3.2 of [1]). To circumvent this difficulty, we used a boundedness on Lorentz space of low-diagonal operator T such that $Tf(t) = \int_0^t k(t,s) f(s)\, ds$.

1.1.2. *Navier-Stokes equations.* I focused on the local existence and uniqueness of strong solution deriving the evolution of fluids with vacuum. The global existence of weak solution of Navier-Stokes equation with vacuum is well-known (see [22] and references therein) but the uniqueness remains still open even for two dimensional case. It has been known that to guarantee the uniqueness, a strong regularity is inevitable. But there has almost not been known for a general fluid with vacuum.
(1) Hence, first we considered two or three dimensional density dependent Navier-Stokes equations for incompressible fluid with vacuum [6]. In general, the regularity of solution is mainly gained by the parabolicity of the momentum equation. But in the presence of the vacuum, the momentum equation looses the parabolicity. Thus we need a compatibility condition which turns out to be necessary and sufficient for a solution to have the strong regularity. Under this condition, we obtained a local existence of strong solution and applied the developed method to a barotropic compressible fluid, which is governed by compressible Navier-Stokes equation with $p = p(\rho)$ [4].

(2) In [7] we further discussed polytropic compressible fluid with vacuum. The most difficult one causing obstacles to strong regularity is the quadratic nonlinearity $|du|^2$ in the energy equation which correlates heavily ρ, u and e. Moreover, the vacuum deepens the correlation. To overcome this complexity, we examined elaborately the correlation and obtained a strong estimate which can be applied to the regularity problem of the barotropic compressible Navier-Stokes equations [8]. The most important results is the local strong solvability which is sharp in view of the recent result [5] where we proved that there is no global strong solution if the initial density has compact support.

1.2. **Harmonic analysis.** I also studied several problems related to Fourier restriction conjecture, Bochner-Riesz conjecture and oscillatory integrals issued in harmonic analysis.

1.2.1. *Bochner-Riesz operator.* We studied the boundedness of Bochner-Riesz operator T^α on $\mathbb{R}^n$, $n \geq 2$, of order α which is a multiplier operator defined by $\widehat{T^\alpha f}(\xi) = \frac{(1-|\xi|^2)_+^\alpha}{\Gamma(\alpha+1)} \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n$, where $\widehat{}$ denotes the Fourier transform, Γ is the gamma function and $r_+ = r$ if $r \geq 0$ and $r_+ = 0$ if $r < 0$. When $\alpha \leq -1$, this definition makes sense by analytic continuation. The problem is to find optimal range of exponents p, q with respect to α satisfying $\|T^\alpha f\|_{L^q} \leq C\|f\|_{L^p}$. We considered the case of negative α and in [10] improved known results by using parabolic re-scaling technique and recent works for bilinear restriction of T. Tao [28]. The techniques could be applied to the boundedness of a cone multiplier operator or an operator defined by convolution of a kernel with singularity over the time-space cone, for example wave operator [11].

1.2.2. *Maximal operator for filtration.* Let $E_n(filtration)$ be measurable subset satisfying $E_n \subset E_{n+1}$ and then define a maximal operator T^* by $T^*f(x) = \sup_n |T(f\chi_{E_n})(x)|$. This operator was used first by M. Christ and A. Kiselev [15, 16] to verify the stability of absolutely continuous spectrum of one dimensional Schrödinger operator $H_V = -d^2/dx^2 + V$. They proved a mapping property of T^* on the Lebesgue space ($L^p \to L^q, q > p$). Observing that this property comes from the convexity difference between the functions t^p and t^q generating the Lebesgue space, we observed in [9] that the maximal inequality holds for Lorentz space ($L^{p,r} \to L^{q,s}(p \leq r < s \leq q)$) and Orlicz space ($L^\Phi \to L^\Psi$ for some function with a convexity difference) containing Lebesgue space naturally. The importance of this mapping property is the application to establishing Strichartz estimates for Schrödinger operator on Lorentz space base [1].

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2. List of Papers

2.1. Published papers.

1. (with E. Koh and S. Lee) *A maximal inequality for filtration on some function spaces*, Osaka J. Math. **41** (2004), 267-276.
2. (with H. J. Choe and H. Kim) *Unique solvability of the initial boundary value problems for compressible viscous fluids*, J. Math. Pures Appl. **83** (2004), 243-275.
3. (with H. Kim) *Unique solvability for the density-dependent Navier-Stokes equations*, Nonlinear Analysis **59** (2004), 465-489.
4. (with Y. Shim) *Weighted L^2 estimates for maximal operators associated to dispersive equations*, Illinois J. Math. **48** (2004), 1081-1092.
5. (with Y. Kim, S. Lee and Y. Shim) *Sharp $L^p - L^q$ estimates for Bochner-Riesz operators of negative index in $\mathbb{R}^n, n \geq 3$*, J. Func. Anal. **218** (2005) 150-167.
6. (with C. Ahn) *Lorentz space extension of Strichartz estimates*, to appear in Proc. AMS.

2.2. Preprints.

1. (with H. Kim) *Existence results for viscous polytropic fluids with vacuum*, Hokkaido Univ. Preprint Series in Math. ♯675, 2004.
2. (with H. Kim) *On the classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities*, Hokkaido Univ. Preprint Series in Math. ♯676, 2004.
3. (with B.J. Jim) *Blow-up of the viscous heat-conducting compressible flow*, Hokkaido Univ. Preprint Series in Math. ♯680, 2004.
4. (with S. Lee, Y. Kim and Y. Shim) *$L^p - L^q$ estimates for convolutions with distribution kernels having singularities on the light cone*, Hokkaido Univ. Preprint Series in Math. ♯684, 2005.
5. (with S. Lee and Y. Shim) *A maximal inequality associated to Schorödinger type equation*, Hokkaido Univ. Preprint Series in Math. ♯685, 2005.
6. (with Y. Shim) *Global estimates of maximal operators generated by dispersive equations*, Hokkaido Univ. Preprint Series in Math. ♯704, 2005.

3. List of Presentations

1. *Asymptotic behavior of Nonlinear Schrödinger equation*, Academic Seminar, KIAS, Korea, 2002.
2. *Unique solvability of the initial boundary value problems for compressible viscous fluids*, International Conference on Nonlinear PDE and Related Topics: Celebrating Neil Trudinger's 60th Birthday, Australian National University, Canberra, 2002.
3. *Local existence for viscous polytropic fluid with vacuum*, The 1st PDE Workshop of Educational Science Institute, Cheju National University, Korea, 2003.
4. *Existence result of semilinear Schrdinger equation*, Lectures on Analysis, Dept. Math. Ajou University, Korea, 2003.
5. *Local existence for heat-conducting incompressible fluid*, The 6th Workshop on Differential Equations, Mathematical Research Center, Chonnam National University, Korea, 2003.
6. *Regularity results for viscous compressible fluids with vacuum*, The 12th Applied Mathematical Forum, Korea, 2004.
7. *Blow-up of the viscous heat-conducting compressible flow*, The 6th Northeastern Symposium on Mathematical Analysis, Tohoku University, Japan, 2005.