COE Postdoctoral Research Report

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1 Quantum cohomology of projective bundles and Hirzebruch surfaces.

Our most recent results give an answer to the following question:

Problem 2: *Classify the quantum cohomology rings of the Hirzebruch surfaces* $F_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))$, $n \geq 0$, *and the projective bundles* $G_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k) \oplus \mathcal{O}(-2-k))$, $k \geq -1$, *by using mirror symmetry.*

Note that we can of course use a direct localization calculation in order to determine the quantum cohomology rings of these spaces. Yet, since localization can be unwieldy, it is advantageous to be able to apply the techniques of mirror symmetry.

We begin by noting some of the cases which have already been solved in the literature. For the Hirzebruch surfaces F_n for $n = 0, 1, 2$, there are a variety of standard approaches which can be applied. Let $QH^*(X)$ denote the quantum cohomology ring of X. It is known, in particular, that $QH^*(F_0) \cong QH^*(F_2)$. For the projective bundles G_k, Givental has shown that $QH^*(G_{-1}) \cong QH^*(G_0)$, i.e. that $\mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2))$ and $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1))$ have the same quantum cohomology rings.

Our work therefore lies in the cases F_n for $n > 2$ and G_k for $k > 0$. The results go as follows:

Solution 2: *There are two isomorphism types of* $QH^*(F_n)$, *depending on whether* n *is odd or even. There is only one isomorphism type of* $QH^*(G_k)$.

Strategy (for G_k):

1) Perform the Birkhoff factorization of the I *function of* G_k, *in order to recover the* J *function*

2) Change coordinates of the J *function via the mirror map, and then show that this resulting function is equal to the* I *function for* G_{-1}.

2 Equivariant mirror symmetry for $\mathcal{O}(k) \oplus \mathcal{O}(-2-k) \to \mathbb{P}^1$

We now generalize the above approach to include all $X_k = \mathcal{O}(k) \oplus \mathcal{O}(-2-k) \to \mathbb{P}^1$, $k \geq 1$. It is indeed possible to derive a version of mirror symmetry, in the sense that we have a mirror map and a double logarithmic function which reproduces known Gromov-Witten invariants. Comparing to the X_0 case, it turns out that $k \geq 1$ forces us to use the Birkhoff factorization to correctly compute invariants. We also find that the result is a collection of rational functions in the equivariant weights, so that we can only read off enumerative information by specifying values for the weights.

We consider equivariant Gromov-Witten theory on X_k, endowed with a T^2 action with weights (λ_1, λ_2) on the respective bundle factors $\mathcal{O}(k) \oplus \mathcal{O}(-2-k)$. Then Givental tells us that mirror symmetry for X_k should be encoded in the following hypergeometric function:

$$I_k^T = e^{p \log q/\hbar} \sum_{d \geq 0} \frac{\prod_{m=(-2-k)d+1}^{0} ((-2-k)p + m\hbar + \lambda_2)}{\prod_{m=1}^{d} (p + m\hbar)^2 \prod_{m=1}^{kd} (kp + m\hbar + \lambda_1)} q^d = e^{p \log q/\hbar} \sum_{d \geq 0} C_d^k(\lambda_1, \lambda_2) q^d. \tag{2.1}$$

While this may be the correct I function, in its given form it is not clear exactly how one should extract Gromov-Witten data from it. For example, if $k = 1$, then in the nonequivariant limit $\lambda_1 = \lambda_2 = 0$, this function reduces to $I_{K_{\mathbb{P}^2}}$, the I function for $\mathcal{O}(-3) \to \mathbb{P}^2$, whose invariants we are not presently interested in. Our guiding principle at the moment is that we would like to reproduce the famous multiple cover formula for curves which is known from calculations on X_{-1}.

There is a way around these difficulties, which goes as follows. We first expand the coefficients $C_d^k(\lambda_1, \lambda_2)$ in powers of $1/\lambda_1$. This has the unfortunate effect of introducing positive powers of $\hbar$ into the expansion of I_k^T. However, from our work in the previous sections, we know that we can eliminate positive powers of $\hbar$ by using Birkhoff factorization. Call the result of this J_k^T. Then the surprising fact is that J_k^T actually contains exactly the expected mirror symmetry data! In other words, we find

$$J_k^T = 1 + \frac{p(\log q + t^k(q, \lambda_1, \lambda_2)) + \tilde{t}^k(q, \lambda_1, \lambda_2)}{\hbar} + \frac{pW^k(q, \lambda_1, \lambda_2) + \tilde{W}^k(q, \lambda_1, \lambda_2)}{\hbar^2} + \dots \tag{2.2}$$

Let $q^k(t^k)$ be the inverse of $\log q + t^k(q, \lambda_1, \lambda_2)$. Then if we specialize equivariant weights so that $\lambda_1 = \lambda_2 = \lambda$, we arrive at

$$W^k(q^k(t^k), \lambda) = -2\lambda Li_2(e^t) \tag{2.3}$$

which is exactly the multiple cover formula. This agrees with known results for the diagonal torus action $\lambda_1 = \lambda_2$.

Publications

1) Brian Forbes and Masao Jinzenji, *Prepotentials for local mirror symmetry via Calabi-Yau fourfolds.* 42 pages. To appear in JHEP. Accepted March, 2006. arXiv: hep-th/0511005.

2) Brian Forbes and Masao Jinzenji, *Extending the Picard-Fuchs system of local mirror symmetry.* J. Math. Phys. 46, 082302 (2005), pp 1-39. arXiv: hep-th/0503098.

3) Brian Forbes, *Computations on B-model geometric transitions.* Mod. Phys. Lett. A, Vol. 20, No. 35 (2005), pp 2685-2697. arXiv: hep-th/0408167.

4) Brian Forbes, *Open string mirror maps from Picard-Fuchs equations.* "Mirror Symmetry V", AMS/IP Advanced Studies in Mathematics Series, to appear. 11 pages. Accepted November, 2004.

5) Brian Forbes, *Open string mirror maps from Picard-Fuchs equations on relative cohomology.* Ph.D. Thesis, University of California, Los Angeles. arXiv:hep-th/0307167. 24 pages. Accepted June, 2004.

Lectures

New examples in mirror symmetry, given at UCLA, Los Angeles, California. March, 2006.

New examples in mirror symmetry, given at the Tokyo Metropolitan University, Tokyo, Japan. February 2006.

Compactifications and Local Mirror Symmetry, given at the Summer Institite for String Theory 2005, Sapporo Garden Palace Hotel. Sapporo, Hokkaido, Japan. August, 2005.

Toric Mirror Symmetry, a series of three lectures given (in Japanese) for the Hokkaido University COE Lecture Series. Hokkaido University, Sapporo, Japan. August, 2005.

Extending the Picard-Fuchs system of local mirror symmetry, given at Tokyo University, Komaba. Tokyo, Japan. January, 2005.

Extending the Picard-Fuchs system of local mirror symmetry, given at "Quantum Cohomology and Mirror Symmetry Day". Tokyo Metropolitan University. Tokyo, Japan. January, 2005.

Computations on B model geometric transitions, given at the "Mirror Workshop: Homological Geometry and Mirror Symmetry." Presented by the COE program of Nagoya University. Fukui, Japan. September, 2004.

Computations on B model geometric transitions, given at the Hangzhou-Beijing International Summer School in Mathematical Physics. Center of Mathematical Sciences, Zhejiang University, Hangzhou, China. July, 2004.

Open String Mirror Maps from Picard-Fuchs equations, given at "Calabi-Yau Varieties and Mirror Symmetry." Banff International Research Station. Banff, Alberta, Canada. December, 2003.